Transport properties in disordered ratchet potentials

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The role of disorder in one-dimensional ratchet potentials is investigated by introducing the following: (i) *impurities*, where a certain fraction of the asymmetric unit cells are replaced by unit cells with opposite asymmetry, and (ii) *randomness*, where all unit cells have the same asymmetry, but random size. The relevant color-induced currents are determined and compared with the current of the ideal periodic ratchet potential. Altogether disorder is shown to quench the effectiveness of thermal ratchets, while remarkable transport properties become detectable. [S1063-651X(97)09708-0]

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Thermal ratchets are remarkable devices capable of rectifying a zero-mean noisy signal [1–9]. The simplest example of a thermal ratchet is described by the stochastic process

$$\dot{x}=-V'(x)+\xi(t), \tag{1}$$

where $\xi(t)$ denotes a Gaussian, zero-mean, stationary noise and $V(x)$ is a periodic potential with unit-cell length L, that is, $V(x+L)=V(x)$. A stationary nonzero average probability current $j(\tau)\equiv(1/L)\int_0^L j(x;\tau)dx$ may result from the combined action of the *spatial asymmetry* of the drift $V(x)\neq V(-x)$ and the noise *finite correlation* with time τ. The process (1) represents a nonequilibrium dynamics, so that no violation of the second law of thermodynamics is implied [10]. Ratchets have been proposed to rectify *periodic* signals too (rocked ratchets [11]).

The purpose of the present paper is to investigate the role of *disorder* in the transport properties of a thermal ratchet. Two instances of disorder are discussed in some detail: (i) *impurities*, for which a certain fraction of the asymmetric unit cells (or *teeth*) of the potential $V(x)$ are replaced at random by unit cells with opposite asymmetry (Fig. 1), and (ii) *randomness*, for which the potential teeth all have the same asymmetry, but their size is randomly distributed. A variety of disordered ratchets may be construed by combining disorder of types (i) and (ii). We conclude that disorder quenches the rectifying power of both thermal and rocked ratchets; most notably, a number of ratchet-related transport properties become detectable in the presence of disorder. Our results allow us to extend the notion of thermal ratchet to the model of transport in polymers [12] and biological macromolecules [3,9], on random surfaces (interfaces) [13], in arrays of Josephson junctions [14], of vortex lines in two-dimensional superconductors [15], and of point defects and dislocations in polycrystalline media [16], to quote but a few examples.

Let us specialize the ratchet model (1) by choosing the autocorrelation function

$$\langle\xi(t)\xi(0)\rangle=\sigma^2\exp(-|t|/\tau), \tag{2}$$

with $\sigma^2=D/\tau$ for the noise source $\xi(t)$ and the shape

$$V(x)=\begin{cases} V_i^m+A_1(x-x_i), & 0\leqslant x-x_i\leqslant l_{1,i} \\ V_{i+1}^m-A_2(x-x_{i+1}), & -l_{2,i}\leqslant x-x_{i+1}\leqslant 0 \end{cases} \tag{3}$$

for the ith ratchet tooth (see the inset of Fig. 1). The two adjacent potential minima $V_i^m=V(x_i)$ and $V_{i+1}^m=V(x_{i+1})$ are separated by a distance $x_{i+1}-x_i=l_{1,i}+l_{2,i}$ and a potential maximum $V_i^M=V_i^m+A_1l_{1,i}=V_{i+1}^m+A_2l_{2,i}$. The overall potential function $V(x)$ is fully determined when the sequence of the minima V_i^m or, equivalently, the sequence of the lengths $l_{1,i}$ and $l_{2,i}$ for a given choice of V_0^m – is assigned. The ideal *periodic* ratchet potential [4–6] corresponds to setting $V_i^m=0$ for $i=0,\pm1,\pm2,\ldots$, so that $l_{1,i}=l_1$ and $l_{2,i}=l_2$ with $l_1+l_2=L$ and $V_i^M=V^M$ with $V^M=A_1l_1=A_2l_2$.

The directionality of the ratchets in Fig. 1 is determined by the choice of A_1 and A_2; here A_1 and A_2 are independent of the tooth index i and $A_2>A_1$. Thus we define the dimensionless *rectifying factor*

$$\gamma(\tau,D)=2\frac{\mu_+(\tau,D)-\mu_-(\tau,D)}{\mu_+(\tau,D)+\mu_-(\tau,D)}, \tag{4}$$

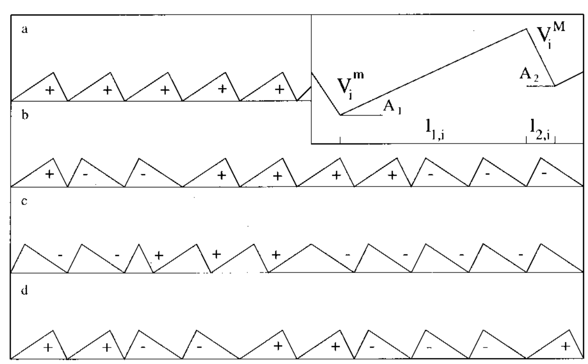

FIG. 1. Ratchet potentials $V(x)$: (a) periodic, made of + cells (i.e., $A_2>A_1$); (b) and (c) with impurities represented by − cells. Impurities in (b) and (c) are topologically different. The potential in (c) can be obtained from the one in (b) through the discrete transformation $V(x)\rightarrow -V(-x)$. A +,− interface divides a + cell sequence on the right-hand side from a − cell sequence on the left-hand side (vice versa for a −,+ interface). (d) Periodic pattern of ± cells. The extended unit cell is made here of two + and two − cells. Inset: the ratchet tooth of Eq. (3).

where $\mu_\pm$ are the escape rates for any transition $x_i \rightarrow x_{i\pm 1}$, respectively. For the piecewise linear potentials of Eq. (3), the factor $\gamma(\tau,D)$ is independent of the size of the ratchet teeth; the relevant escape rates $\mu_\pm(\tau,D)$ have been calculated in the strong color limit of Ref. [17], whence $\mu_\pm(0,D) \equiv \mu_0(D)$ and for $A_2 - A_1 \ll \sigma$

$$\gamma(\tau,D) \simeq \frac{\tau}{D} \frac{A_2^3 - A_1^3}{A_2 + A_1}. \tag{5}$$

Notice that in the weak color limit of Refs. [4–6], the rectifying factor $\gamma(\tau,D)$ would be proportional to τ^2. The linear τ dependence in Eq. (5), an artifact of the discontinuities of the drift factor $V'(x)$, has no bearing on the conclusions of the present work. The stationary average current $j(\tau)$ is proportional to $\gamma(\tau,D)$ [4–6].

(i) Disorder from impurities. Let us assume that the periodic ratchet potential $V_i^m = 0$ and $l_1 + l_2 = L$ is perturbed by adding impurities at random. An impurity is modeled here as a unit cell with the same length L but reversed asymmetry $A_1 \leftrightarrow A_2$ (and reversed rectifying factor). Impurities (or $-$ cells) can replace the ratchet teeth (or $+$ cells) in two topologically different ways, as illustrated in Figs. 1(b) and 1(c) [18].

We focus here on $V(x)$ configurations where the $\pm$ cells are ordered to form random spatial sequences distributed according to Poissonian distributions [19] with characteristic lengths $\nu_+ L$ and $\nu_- L$, respectively, and $\nu_\pm \gg 1$. The stationary average current $j(\tau)$ (in the absence of periodicity the average is taken over the entire ratchet length) can be easily calculated by adding the contribution from each $\pm$ cell, that is [19],

$$j(\tau) = j_0(\tau) \frac{\nu_+ - \nu_-}{\nu_+ + \nu_-}, \tag{6}$$

where $j_0(\tau) = L \mu_0(D) \gamma(\tau,D)$ is the current in the unperturbed ratchet potential (made of $+$ cells).

In homogeneous random potentials with $\nu_- = \nu_+$, the *local* probability current $j(x;\tau)$ is spatially correlated, even if its mean in Eq. (6) vanishes. Suppose that a Brownian particle is injected in the unit cell located at x_0; then the current $j(x_0, \tau)$ would be either positive or negative depending on the distribution of the $\pm$ cells in the neighborhood of x_0. As a consequence, the particle would travel a certain distance Δx from x_0, the current changing randomly with the position of the particle. On average, the current decays exponentially with Δx [19], that is, $j(x_0 + \Delta x; \tau) = j(x_0; \tau) \exp(-|\Delta x|/\lambda)$, where $\lambda = \nu_\pm L/2$ with $\lambda \gg L$. This means that the injected particle would drift in either direction according to the logarithmic law [20]

$$\langle \Delta x(t) \rangle \sim \lambda \ln[\mu_0 \gamma(\tau,D) t]. \tag{7}$$

For the sequence of $\pm$ cells of Fig. 1(b) with $\nu_+ = \nu_-$ the $+,-$ interfaces are natural accumulation points, whereas the $-,+$ interfaces are rather depletion points. The question then arises under what circumstances a $+,-$ interface acts as a stable accumulation (or *localization*) point. On both sides of the stable $+,-$ interface, the total probability per unit cell decreases by a factor $e^{\gamma(\tau,D)}$ on moving away from the interface itself. This is required to compensate for the difference $\mu_+(\tau,D) - \mu_-(\tau,D)$ and guarantee that $j(x;\tau)$ vanishes at the interface. Stability of the accumulation point $+,-$ is thus achieved when the accumulation length $\lambda_l = L/\gamma(\tau,D)$ is much shorter than the distance l between the $+,-$ interface and the closer $-,+$ interface, i.e., for $l \gg \lambda_l$, $(1/\lambda)\exp[-(l-\lambda_l)/\lambda]$ [19].

(ii) Disorder from randomness. We assume now that the potential $V(x)$ is made of individual ratchet teeth (3) with constant, positive asymmetry $A_2 > A_1$, but irregular size. This means that the potential minima V_i^m and maxima V_i^M are distributed at random and so are the distances $l_{1,i}$ (between V_i^m and V_i^M) and $l_{2,i-1}$ (between V_{i-1}^M and V_i^m). We denote the distributions of V_i^m and V_i^M by $\rho(V^m)$ and $\rho(V^M)$, respectively. The distribution function $\rho(\Delta)$ for the barrier heights $\Delta_i = V_i^M - V_i^m$ is uniquely determined by the knowledge of $\rho(V^m)$ and $\rho(V^M)$. The averages $\langle l_1 \rangle$ of $l_{1,i}$ and $\langle l_2 \rangle$ of $l_{2,i}$ are related to $\langle \Delta \rangle$ by $\langle l_1 \rangle = \langle \Delta \rangle / A_1$ and $\langle l_2 \rangle = \langle \Delta \rangle / A_2$; correspondingly, the average ratchet tooth size is $\langle L \rangle = \langle l_1 \rangle + \langle l_2 \rangle$. Notice that all ratchet teeth are characterized by the very same rectifying factor $\gamma(\tau,D)$ of Eq. (5).

For δ-correlated noises, Eq. (2) with $\tau = 0$, the diffusion process in the random potential $V(x)$ above (no drift here) is well described by the mean first-passage time $T(x)$ for a Brownian particle to diffuse, say, from 0 up to x, namely [19],

$$T(x) = \frac{1}{D} \int_0^x \frac{dy}{p(y)} \int_{-\infty}^y p(z) dz, \tag{8}$$

where $p(x) = \mathcal{N} \exp[-V(x)/D]$ and $\mathcal{N}$ is a suitable normalization constant. On assuming for simplicity that $x = N\langle L \rangle$ with $N \gg 1$, we recast Eq. (8) as

$$T(N) \simeq \frac{N^2 \langle L \rangle^2}{2D} \frac{\langle e^{V^M/D} \rangle \langle e^{-V^m/D} \rangle}{\langle \Delta/D \rangle^2} \langle 1 - e^{-\Delta/D} \rangle^2, \tag{9}$$

where $\langle \ \rangle$ denotes the averages with respect to $\rho(V^m)$, $\rho(V^M)$, or $\rho(\Delta)$ as appropriate.

In the presence of correlated noise $\tau > 0$, the ratchet average current $j(\tau)$ can be expressed in terms of the factor $\gamma(\tau,D)$ [4–6] as

$$j(\tau) = \langle L \rangle \gamma(\tau,D) / T(1), \tag{10}$$

whence our prediction

$$j(\tau) = \frac{2D\gamma(\tau,D)}{\langle L \rangle} \frac{\langle \Delta/D \rangle^2}{\langle e^{V^M/D} \rangle \langle e^{-V^m/D} \rangle \langle 1 - e^{-\Delta/D} \rangle^2}. \tag{11}$$

In the following we will discuss this important result in some detail. Here we note that for noise intensities larger than the standard deviation σ_V of V_i^m, $D \gg \sigma_V$, the current (11) approaches $2D\gamma(\tau,D)/\langle L \rangle$, no matter what the distributions of V_i^m and Δ_i.

We apply now our prediction (11) for $j(\tau)$ to a few examples of the random potential $V(x)$.

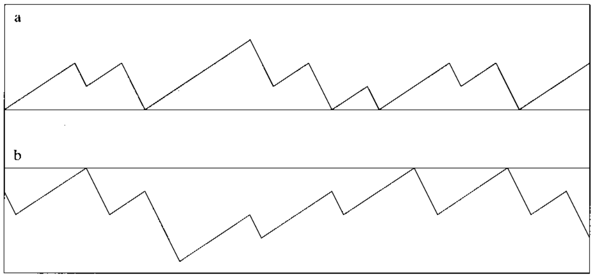

FIG. 2. Random ratchet potentials $V(x)$: (a) exponentially distributed minima with $V_i^m \geqslant 0$ and (b) exponentially distributed minima with $V_i^m \leqslant 0$ (see the text).

(a) Periodic with $V_i^m = 0$ *and* $\Delta_i = V^M$. This is the standard ratchet model [4–6] discussed in item (i) (with $\nu_- = 0$). Equation (11) leads immediately to the current $j_0(\tau)$ of Eq. (6).

(b) Exponentially distributed minima with $V_i^m \geqslant 0$. This is the case illustrated in Fig. 2(a). The relevant distributions for V_i^m and Δ_i are $\rho(V^m) = (1/\sigma_V)\exp(-V^m/\sigma_V)$ and $\rho(\Delta) = (1/\sigma_V)\exp(-\Delta/\sigma_V)$, respectively. These are multistable potentials with degenerate minima $V_i^m = 0$. An explicit calculation proves that $\langle e^{V^M/D} \rangle \langle e^{-V^m/D} \rangle$ converges to $D^2/(D^2 - \sigma_V^2)$ for $\sigma_V < D$ and diverges otherwise. Accordingly,

$$j(\tau) = \frac{2D\gamma(\tau,D)}{\langle L \rangle}\left[1 - \left(\frac{\sigma_V}{D}\right)^4\right] \tag{12}$$

for $\sigma_V < D$ and $j(\tau) = 0$ for $\sigma_V > D$. This leads us to conclude that for $D < \sigma_V$ the roughness of $V(x)$ cannot be overcome, no matter what the noise correlation time; again, disorder quenches the rectifying power of the ratchet. Equivalently, on keeping the noise variance σ^2 in Eq. (2) fixed and tuning τ as a control parameter, a threshold τ_{th} can be introduced, so that $j(\tau) \sim \tau - \tau_{th}$ for $\tau \to \tau_{th}+$ and $j(\tau) = 0$ for $\tau \leqslant \tau_{th}$.

(c) Exponentially distributed minima with $V_i^m \leqslant 0$. This class of potential can be obtained from the potentials of example (b) by turning the relevant potential upside down and interchanging A_1 and A_2, i.e., by transforming $V(x)$ into $-V(-x)$ as shown in Fig. 2(b). As a consequence, the result of Eq. (12) remains unchanged [21]. As a major difference, however, the potentials in (b) always support stationary diffusion, whereas the potentials in (c) do not: they are not bounded from below and for $D < \sigma_V$ the Brownian particle diffuses to infinity.

(d) Gaussian distributed maxima and minima. Let V_i^m and V_i^M be distributed according to the same Gaussian law $\rho(V) = (2\pi\sigma_V^2)^{-1/2}\exp(-V^2/2\sigma_V^2)$. This implies that $\rho(\Delta)$ is a Gaussian function too, with $\Delta \geqslant 0$ and variance $2\sigma_V^2$. Simple algebraic manipulations yield the approximate expression

$$j(\tau) = \frac{2D\gamma(\tau,D)}{\langle L \rangle} e^{-\sigma_V^2/D^2} \tag{13}$$

for $D \ll \sigma_V$. At variance with examples (b) and (c) the current $j(\tau)$ is definite positive for $D \leqslant \sigma_V$ too, although exponentially suppressed. We conclude item (ii) by noticing that *nonmarginal* ratchet currents in random potentials can be observed only for finite band roughness (the Gaussian distribution being just a borderline case).

Quenched noise helps make ratchet models more realistic. For instance, the directed movement of polymerase (and other DNA or RNA binding proteins) along the nucleotide backbone is certainly controlled by randomness too (quasirandom sequences, folding, etc.) [9]. According to Eq. (12), disorder would become detectable as a *threshold* in the relevant ratchet current $j(T) \sim T - T_{th}$ or $j(\tau) \sim \tau - \tau_{th}$. Moreover, we know that the extension of the above predictions to the transport of (arrays of) directed lines [13] on disordered substrates requires minor conceptual modifications [8]. Remarkable applications to dislocation [16] and flux line dynamics [15] follow immediately. A detailed investigation of quenched disorder in (1+1)-dimensional ratchets is planned to be reported in a forthcoming paper.

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